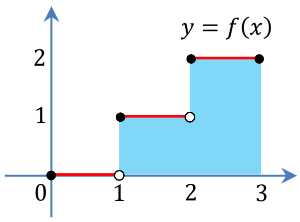
Solution to Problem Set 7

1. (a) The graph of f is as shown below.



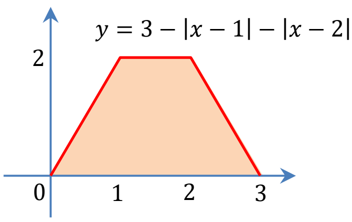
The given integral represents the total area of a unit square and a rectangle with height 2 units and base width 1 unit, i.e.

$$\int_0^3 f(x)dx = 1 \cdot 1 + 2 \cdot 1 = 3.$$

(b) We have

$$3 - |x-1| - |x-2| = \begin{cases} 3 + (x-1) + (x-2) & \text{if } 0 \le x < 1 \\ 3 - (x-1) + (x-2) & \text{if } 1 \le x < 2 \\ 3 - (x-1) - (x-2) & \text{if } 2 \le x < 3 \end{cases} = \begin{cases} 2x & \text{if } 0 \le x < 1 \\ 2 & \text{if } 1 \le x < 2 \\ 6 - 2x & \text{if } 2 \le x < 3 \end{cases},$$

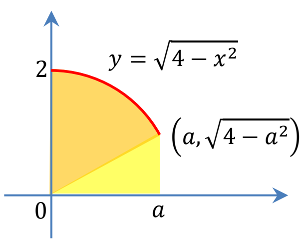
and the graph of the integrand is as shown below.



Referring to the graph, we see that the given integral represents the area of a trapezoid (trapezium) with base widths 1 and 3 units and height 2 units, so

$$\int_0^3 (3 - |x-1| - |x-2|)\, dx = \frac{1+3}{2} \cdot 2 = 4.$$

(c) The graph of the integrand is as shown below.



The given integral represents the total area of a circular sector of angle $\arcsin\frac{a}{2}$ and of radius 2 centered at the origin, together with a right-angled triangle beneath it, as shown in the diagram. So

$$\int_0^a \sqrt{4 - x^2}\, dx = \underbrace{\frac{\arcsin\frac{a}{2}}{2\pi} \cdot \pi(2)^2}_{\text{area of sector}} + \underbrace{\frac{1}{2}(a)\left(\sqrt{4-a^2}\right)}_{\text{area of triangle}} = 2\arcsin\frac{a}{2} + \frac{1}{2}a\sqrt{4-a^2}.$$

2. (a)

$$\int_b^a \cos^2 x\,dx = \int_a^b -\cos^2 x\,dx = \int_a^b (\sin^2 x - 1)dx = \int_a^b \sin^2 x\,dx - \int_a^b dx$$
$$= I - (b-a) = I + a - b$$

(b)

$$\int_a^b \cos 2x\,dx = \int_a^b (1 - 2\sin^2 x)dx = \int_a^b dx - 2\int_a^b \sin^2 x\,dx$$
$$= (b-a) - 2I = b - a - 2I$$

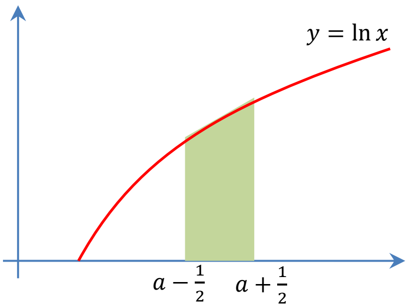
3. Let $f\colon (0,+\infty) \to \mathbb{R}$ be the function $f(x) = \ln x$. Since

$$f''(x) = -\frac{1}{x^2} < 0$$

for every $x > 0$, we observe that the graph of f is concave downward on $(0,+\infty)$. Now let $a \geq 2$ be given.

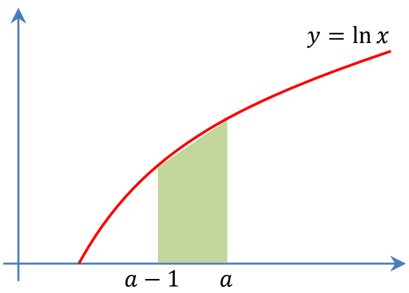
(a) Consider the trapezium bounded by the tangent line to the graph of f at the point $(a, \ln a)$, the x-axis, and the vertical lines $x = a - \frac{1}{2}$ and $x = a + \frac{1}{2}$. The area of this trapezium is $\ln a$. Since the graph of f is concave downward on $\left[a - \frac{1}{2}, a + \frac{1}{2}\right]$, the region represented by $\int_{a-\frac{1}{2}}^{a+\frac{1}{2}} \ln x\,dx$ lies completely inside the trapezium. Therefore

$$\int_{a-\frac{1}{2}}^{a+\frac{1}{2}} \ln x\,dx \leq \ln a.$$



(b) Consider the trapezium with vertices $(a, \ln a)$, $(a-1, \ln(a-1))$, $(a-1, 0)$ and $(a, 0)$. This trapezium has area $\frac{\ln(a-1)+\ln a}{2}$. Since since the graph of f is concave downward on $[a-1, a]$, the trapezium lies completely inside the region represented by $\int_{a-1}^{a} \ln x\,dx$ (cf. Remark 4.76). So

$$\int_{a-1}^{a} \ln x\,dx \geq \frac{\ln(a-1) + \ln a}{2}.$$



4. (a) For every $x \in [0,1]$, we have $x^3 \le x^2$. Since the function $\cos$ is strictly decreasing on $[0,1]$, we have

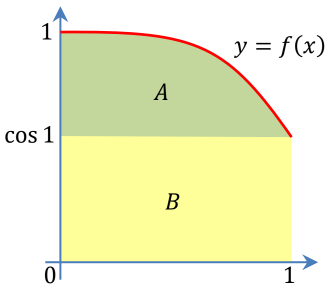
$$\cos(x^3) \ge \cos(x^2) \qquad \text{for every } x \in [0,1],$$

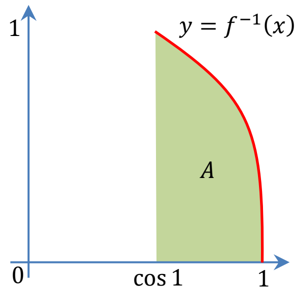
i.e. $g(x) \ge f(x)$ for every $x \in [0,1]$. Therefore

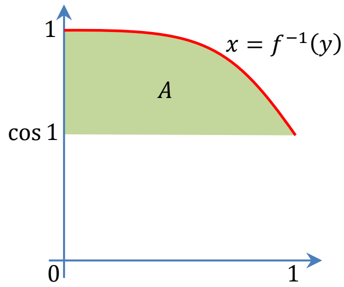
$$\int_0^1 g(x)dx \ge \int_0^1 f(x)dx$$

by integral inequality. Note that the inequality $\ge$ can actually be replaced by a strict inequality $>$ because in fact $x^3 < x^2$ for every $x \in (0,1)$.

(b) Note that the function $f(x) = \cos(x^2)$ is strictly decreasing on $[0,1]$, and recall from Remark 1.61 that the graph of f^{-1} is the "mirror reflection" of the graph of f across the line $y = x$.







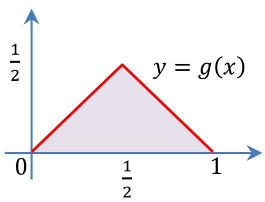
Now refer to the above diagrams with two regions whose areas are A and B. The integral $\int_0^1 f(x)dx$ represents the area $A + B$, while $\int_{\cos 1}^1 f^{-1}(x)dx$ represents the area of a region that is congruent to A only. Therefore

$$\int_0^1 f(x)dx > \int_{\cos 1}^1 f^{-1}(x)dx.$$

Remark: Note that the reasoning in (b) applies not only to the function $f(x) = \cos(x^2)$, but also to any strictly decreasing function with appropriate domain and range.

5. Let $g\colon [0,1] \to [0,+\infty)$ be the piecewise defined function

$$g(x) = \begin{cases} x & \text{if } x \in [0,1/2] \\ 1-x & \text{if } x \in (1/2,1] \end{cases}.$$



Then g is non-negative and continuous on $[0,1]$, and $\int_0^1 g(x)dx$ represents the area of the isosceles triangle with vertices $(0,0)$, $(1,0)$ and $\left(\frac{1}{2},\frac{1}{2}\right)$, so

$$\int_0^1 g(x)dx = \frac{1}{2} \cdot 1 \cdot \frac{1}{2} = \frac{1}{4}.$$

Now by the given condition we have $f(x) \le g(x)$ for every $x \in [0,1]$. So by integral inequality we have

$$\int_0^1 f(x)dx \le \int_0^1 g(x)dx = \frac{1}{4}.$$

■

6. (a) Since f is continuous on $[a,b]$, f must attain global maximum and global minimum on $[a,b]$ according to **Extreme Value Theorem**, i.e. there exist $c_1, c_2 \in [a,b]$ such that

$$f(c_1) \le f(x) \le f(c_2) \qquad \text{for every } x \in [a,b].$$

By integral inequality, we have

$$\int_a^b f(c_1)dx \le \int_a^b f(x)dx \le \int_a^b f(c_2)dx\,,$$

i.e. $f(c_1)(b-a) \le \int_a^b f(x)dx \le f(c_2)(b-a)$. ■

(b) Dividing each component in the inequalities obtained in (a) by the positive number $b-a$, we see that there exist $c_1, c_2 \in [a,b]$ such that

$$f(c_1) \le \frac{1}{b-a}\int_a^b f(x)dx \le f(c_2).$$

Since f is continuous on $[a,b]$, according to the **Intermediate Value Theorem** there exists c between c_1 and c_2 such that

$$f(c) = \frac{1}{b-a}\int_a^b f(x)dx\,,$$

i.e. $\int_a^b f(x)dx = f(c)(b-a)$. Such a number c belongs to (a,b) also. ■

7. (a)

$$\begin{aligned}\int (\tan\theta + \cot\theta)^2 d\theta &= \int (\tan^2\theta + \cot^2\theta + 2\tan\theta\cot\theta)d\theta \\ &= \int [(\tan^2\theta + 1) + (\cot^2\theta + 1)]d\theta \\ &= \int \sec^2\theta\, d\theta + \int \csc^2\theta\, d\theta = \tan\theta - \cot\theta + C,\end{aligned}$$

where C is an arbitrary constant.

(b)

$$\int \frac{1}{\sqrt{(2-2x)(2+2x)}}dx = \frac{1}{2}\int \frac{1}{\sqrt{1-x^2}}dx = \frac{1}{2}\arcsin x + C,$$

where C is an arbitrary constant.

(c)

$$\int 2^{t+2}e^{2t}dt = 4\int (2e^2)^t dt = 4\cdot\frac{(2e^2)^t}{\ln(2e^2)} + C = \frac{2^{t+2}e^{2t}}{2+\ln 2} + C,$$

where C is an arbitrary constant.

(d)

$$\int \frac{x^4}{1+x^2}\,dx = \int \left(x^2 - 1 + \frac{1}{1+x^2}\right) dx = \int x^2 dx - \int dx + \int \frac{1}{1+x^2}\,dx$$

$$= \frac{1}{3}x^3 - x + \arctan x + C,$$

where C is an arbitrary constant.

(e)

$$\int \frac{3+\sin^2 x}{\cos^2 x}\,dx = \int \frac{4-\cos^2 x}{\cos^2 x}\,dx = \int (4\sec^2 x - 1)dx$$

$$= 4\int \sec^2 x\,dx - \int dx$$

$$= 4\tan x - x + C,$$

where C is an arbitrary constant.

(f) On the interval $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$, we have

$$\int \frac{\sin x}{1+\sin x}\,dx = \int \frac{(\sin x)(1-\sin x)}{(1+\sin x)(1-\sin x)}\,dx = \int \frac{\sin x - \sin^2 x}{1-\sin^2 x}\,dx$$

$$= \int \frac{\sin x - \sin^2 x}{\cos^2 x}\,dx = \int (\sec x \tan x - \tan^2 x)dx$$

$$= \int (1 - \sec^2 x + \sec x \tan x)dx$$

$$= x - \tan x + \sec x + C,$$

where C is an arbitrary constant.

8. (a)

$$\frac{d}{d\theta}\left(\frac{1}{3}\tan^3\theta - \tan\theta\right) = \frac{1}{3}\cdot 3\tan^2\theta\sec^2\theta - \sec^2\theta = (\tan^2\theta - 1)\sec^2\theta$$

$$= (\tan^2\theta - 1)(\tan^2\theta + 1) = \tan^4\theta - 1.$$

(b) According to the result from (a), we have

$$\int \tan^4\theta\,d\theta = \left(\frac{1}{3}\tan^3\theta - \tan\theta\right) + \int d\theta$$

$$= \frac{1}{3}\tan^3\theta - \tan\theta + \theta + C,$$

where C is an arbitrary constant.

9. (a) Since $\frac{dy}{dx} = x - k$, we have

$$y = \int (x-k)dx = \frac{1}{2}x^2 - kx + C,$$

where C is some constant. Since $y = 0$ when $x = k$, we have

$$0 = \frac{1}{2}k^2 - k(k) + C,$$

so $C = \frac{1}{2}k^2$. Therefore,

$$y = \frac{1}{2}x^2 - kx + \frac{1}{2}k^2.$$

(b) Since $\frac{du}{dx} = y$, using the result from (a) we have

$$\frac{du}{dx} = \frac{1}{2}x^2 - kx + \frac{1}{2}k^2.$$

Thus,

$$u = \int \left(\frac{1}{2}x^2 - kx + \frac{1}{2}k^2\right) dx = \frac{1}{2}\int x^2 dx - k\int x\,dx + \frac{1}{2}k^2 \int dx$$

$$= \frac{1}{2}\left(\frac{1}{3}x^3\right) - k\left(\frac{1}{2}x^2\right) + \frac{1}{2}k^2x + D = \frac{1}{6}x^3 - \frac{1}{2}kx^2 + \frac{1}{2}k^2x + D,$$

where D is some constant. Since $u = 0$ when $x = k$, we have

$$0 = \frac{1}{6}k^3 - \frac{1}{2}k(k^2) + \frac{1}{2}k^2(k) + D,$$

so $D = -\frac{1}{6}k^3$. Therefore,

$$u = \frac{1}{6}x^3 - \frac{1}{2}kx^2 + \frac{1}{2}k^2x - \frac{1}{6}k^3 = \frac{1}{6}(x^3 - 3kx^2 + 3k^2x - k^3) = \frac{1}{6}(x-k)^3.$$

10. (a) Let $x(t)$, $v(t)$ and $a(t)$ denote respectively the position, velocity and acceleration of the particle after t seconds. Now $a(t) = k - 2t$, so

$$v(t) = \int a(t)dt = \int (k - 2t)dt = kt - t^2 + C,$$

where C is some constant. Since $v(0) = 0$ and $v(2) = 12$, we find that

$$\begin{cases} k \cdot 0 - 0^2 + C = 0 \\ k \cdot 2 - 2^2 + C = 12 \end{cases}$$

so $C = 0$ and $k = 8$.

(b) We have $v(t) = 8t - t^2 = t(8-t)$, so $v(t) = 0$ only when $t = 0$ or $t = 8$. Since

$$v(t) \begin{cases} > 0 & \text{if } 0 < t < 8 \\ < 0 & \text{if } t > 8 \end{cases},$$

the particle will change its direction of motion right after 8 seconds.

(c) We have

$$x(t) = \int v(t)dt = \int (8t - t^2)dt = 4t^2 - \frac{1}{3}t^3 + D,$$

where D is some constant. Since $x(0) = 0$, we have $4(0)^2 - \frac{1}{3}(0)^3 + D = 0$, so $D = 0$ as well. Thus,

$$x(t) = 4t^2 - \frac{1}{3}t^3.$$

If the particle returns to its original position, then we have $x(t) = 0$, i.e. $4t^2 - \frac{1}{3}t^3 = 0$. This gives $t = 0$ or $t = 12$. So the particle will return to its original position after 12 seconds.

11. Let $x(t)$ and $v(t)$ denote respectively the position and velocity of the particle t seconds after it starts moving. Now the constant acceleration of the particle is $-2\ \mathrm{m/s^2}$, so

$$v(t) = \int -2\,dt = -2t + C,$$

where C is some constant.

Since the initial speed is $16\ \mathrm{m/s}$, we have $v(0) = -2(0) + C = 16$, so $C = 16$. Thus

$$x(t) = \int v(t)dt = \int (-2t + 16)dt = -t^2 + 16t + D,$$

where D is some constant. But $x(0) = 0$ gives $D = 0$.

After the particle travels exactly 48 metres, we have $x(t) = -t^2 + 16t = 48$, so

$$t^2 - 16t + 48 = 0$$
$$(t - 4)(t - 12) = 0,$$

giving $t = 4$ or $t = 12$.

At $t = 4$, we have $v(4) = -2(4) + 16 = 8$; at $t = 12$, we have $v(12) = -2(12) + 16 = -8$. At both instants, the speed of the particle is $8\ \mathrm{m/s}$.

12. Since $\frac{d^2y}{dx^2} = 6$, taking antiderivatives twice we have

$$\frac{dy}{dx} = \int 6\,dx = 6x + C \qquad \text{and} \qquad y = \int (6x + C)dx = 3x^2 + Cx + D,$$

where C and D are some constants.

Since y attains a minimum value of 5 at $x = 2$, we have $y(2) = 5$ and $\left.\frac{dy}{dx}\right|_{x=2} = 0$. Thus we have

$$\begin{cases} 3(2)^2 + C(2) + D = 5 \\ 6(2) + C = 0 \end{cases},$$

so $C = -12$ and $D = 17$. Therefore the equation of the curve is given by

$$y = 3x^2 - 12x + 17.$$

13. (a) The area of the shaded region is given by

$$\int_0^{\pi} \sin x\, dx = [-\cos x]_0^{\pi} = (-\cos \pi) - (-\cos 0) = 2.$$

(b) Since the area under the graph f between $x = a$ and $x = \pi - a$ is half of the whole shaded area, this part of the region has an area of 1 squared unit. Thus,

$$1 = \int_a^{\pi - a} \sin x\, dx = [-\cos x]_a^{\pi - a} = [-\cos(\pi - a)] - (-\cos a) = 2\cos a.$$

Since $a \in \left(0, \frac{\pi}{2}\right)$, this implies that $a = \arccos\frac{1}{2} = \frac{\pi}{3}$, and so

$$b = \sin a = \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}.$$

14. (a) Since $t^3 - 3t^2 + 2t = t(t-1)(t-2)\begin{cases} \geq 0 & \text{if } t \in [0,1] \cup [2, +\infty) \\ < 0 & \text{if } t \in (-\infty, 0) \cup (1, 2) \end{cases}$, we have

$$\begin{aligned}
&\int_{-1}^{3} |t^3 - 3t^2 + 2t| dt \\
&= -\int_{-1}^{0} (t^3 - 3t^2 + 2t)dt + \int_0^1 (t^3 - 3t^2 + 2t)dt - \int_1^2 (t^3 - 3t^2 + 2t)dt + \int_2^3 (t^3 - 3t^2 + 2t)dt \\
&= -\left[\frac{1}{4}t^4 - t^3 + t^2\right]_{-1}^{0} + \left[\frac{1}{4}t^4 - t^3 + t^2\right]_0^1 - \left[\frac{1}{4}t^4 - t^3 + t^2\right]_1^2 + \left[\frac{1}{4}t^4 - t^3 + t^2\right]_2^3 \\
&= -\left(0 - \frac{9}{4}\right) + \left(\frac{1}{4} - 0\right) - \left(0 - \frac{1}{4}\right) + \left(\frac{9}{4} - 0\right) = 5
\end{aligned}$$

(b)

$$\begin{aligned}
\int_0^{\frac{\pi}{4}} \frac{1 - \sin^3\theta}{\cos^2\theta} d\theta &= \int_0^{\frac{\pi}{4}} \frac{1 - \sin\theta\,(1 - \cos^2\theta)}{\cos^2\theta} d\theta = \int_0^{\frac{\pi}{4}} \left(\frac{1}{\cos^2\theta} - \frac{\sin\theta}{\cos^2\theta} + \sin\theta\right) d\theta \\
&= \int_0^{\frac{\pi}{4}} (\sec^2\theta - \sec\theta\tan\theta + \sin\theta) d\theta = [\tan\theta - \sec\theta - \cos\theta]_0^{\frac{\pi}{4}} \\
&= \left(1 - \sqrt{2} - \frac{\sqrt{2}}{2}\right) - (0 - 1 - 1) = 3 - \frac{3\sqrt{2}}{2}
\end{aligned}$$

(c)

$$\begin{aligned}
\int_1^{\sqrt{3}} \frac{1 + x + x^2}{x + x^3} dx &= \int_1^{\sqrt{3}} \left(\frac{1 + x^2}{x + x^3} + \frac{x}{x + x^3}\right) dx \\
&= \int_1^{\sqrt{3}} \left(\frac{1}{x} + \frac{1}{1 + x^2}\right) dx = [\ln x + \arctan x]_1^{\sqrt{3}} \\
&= \left(\ln\sqrt{3} + \frac{\pi}{3}\right) - \left(0 + \frac{\pi}{4}\right) = \frac{1}{2}\ln 3 + \frac{\pi}{12}.
\end{aligned}$$

15. (a) For each positive integer k, we have $\frac{1}{(k+1)^p} \leq \frac{1}{x^p} \leq \frac{1}{k^p}$ for every $x \in [k, k+1]$. By the integral inequality,

$$\int_k^{k+1} \frac{1}{(k+1)^p}\,dx \leq \int_k^{k+1} \frac{1}{x^p}\,dx \leq \int_k^{k+1} \frac{1}{k^p}\,dx\,,$$

and so

$$\frac{1}{(k+1)^p} \leq \int_k^{k+1} \frac{1}{x^p}\,dx \leq \frac{1}{k^p}.$$

Now for every positive integer n, we sum each component from $k = 1$ to $k = n$. This gives

$$\sum_{k=1}^{n} \frac{1}{(k+1)^p} \leq \sum_{k=1}^{n} \int_k^{k+1} \frac{1}{x^p}\,dx \leq \sum_{k=1}^{n} \frac{1}{k^p}$$

which is the same as

$$\sum_{k=2}^{n+1} \frac{1}{k^p} \leq \int_1^{n+1} \frac{1}{x^p}\,dx \leq \sum_{k=1}^{n} \frac{1}{k^p}.$$

(b) (i) For each positive integer n, using the right inequality of the result from (a) with $p = 1$, we get

$$\sum_{k=1}^{n} \frac{1}{k} \geq \int_1^{n+1} \frac{1}{x}\,dx = [\ln x]_1^{n+1} = \ln(n+1) - \ln 1 = \ln(n+1).$$

(ii) For each positive integer n, using the left inequality of the result from (a) with $p = 2$, we get

$$\sum_{k=2}^{n+1} \frac{1}{k^2} \leq \int_1^{n+1} \frac{1}{x^2}\,dx = \left[-\frac{1}{x}\right]_1^{n+1} = 1 - \frac{1}{n+1}.$$

Thus,

$$\sum_{k=1}^{n} \frac{1}{k^2} = 1 + \sum_{k=2}^{n+1} \frac{1}{k^2} - \frac{1}{(n+1)^2} \leq 2 - \frac{1}{n+1} - \frac{1}{(n+1)^2} < 2.$$

16. (a) Let $u = \csc x + \cot x$. Then $du = -(\csc x \cot x + \csc^2 x)dx$. Thus

$$\int \csc x\,dx = \int \frac{\csc x\,(\csc x + \cot x)}{\csc x + \cot x}\,dx = -\int \frac{1}{u}\,du$$

$$= -\ln|u| + C = -\ln|\csc x + \cot x| + C,$$

where C is an arbitrary constant.

(b) Let $u = e^x$. Then $du = e^x dx$. Thus

$$\int \frac{1}{e^x + e^{-x}}\,dx = \int \frac{e^x}{e^{2x} + 1}\,dx = \int \frac{1}{u^2 + 1}\,du$$

$$= \arctan u + C = \arctan(e^x) + C,$$

where C is an arbitrary constant.

(c)

$$\int \frac{\cos^5\theta}{\sin^7\theta}\,d\theta = \int \frac{\cos^5\theta}{\sin^5\theta}\csc^2\theta\,d\theta = -\int \cot^5\theta\,d\cot\theta$$

$$= -\frac{1}{6}\cot^6\theta + C,$$

where C is an arbitrary constant.

(d)

$$\int \frac{[\ln(u^2)]^2}{u}\,du = \int (2\ln|u|)^2\cdot\frac{1}{u}\,du = 4\int (\ln|u|)^2 d\ln|u| = \frac{4}{3}(\ln|u|)^3 + C,$$

where C is an arbitrary constant.

<u>*Alternative Solution*</u>: Let $t = \ln(u^2)$. Then $dt = \frac{2}{u}du$, so

$$\int \frac{[\ln(u^2)]^2}{u}\,du = \frac{1}{2}\int [\ln(u^2)]^2\cdot\frac{2}{u}\,du = \frac{1}{2}\int t^2 dt = \frac{1}{6}t^3 + C = \frac{1}{6}[\ln(u^2)]^3 + C,$$

where C is an arbitrary constant.

(e) Let $u = x^{\frac{3}{2}}$. Then $x^3 = u^2$ and $du = \frac{3}{2}x^{\frac{1}{2}}dx = \frac{3}{2}\sqrt{x}dx$, so

$$\int \frac{\sqrt{x}}{1+x^3}\,dx = \int \frac{1}{1+u^2}\cdot\frac{2}{3}\,du = \frac{2}{3}\int \frac{1}{1+u^2}\,du$$

$$= \frac{2}{3}\arctan u + C = \frac{2}{3}\arctan\left(x^{\frac{3}{2}}\right) + C,$$

where C is an arbitrary constant.